

Examples of nearly integrable systems on \mathbb{A}^3 with asymptotically dense projected orbits

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Abstract

Given an integer $\kappa \geq 2$, we introduce a class of nearly integrable systems on \mathbb{A}^3 , of the form

$$H_n(\theta, r) = \frac{1}{2}\|r\|^2 + \frac{1}{n}U(\theta_2, \theta_3) + f_n(\theta, r)$$

where $U \in C^\kappa(\mathbb{T}^2)$ is a generic potential function and f_n a $C^{\kappa-1}$ additional perturbation such that $\|f_n\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \frac{1}{n}$, so that H_n is a perturbation of the completely integrable system $h(r) = \frac{1}{2}\|r\|^2$.

Let $\Pi : \mathbb{A}^3 \rightarrow \mathbb{R}^3$ be the canonical projection. We prove that for each $\delta > 0$, there exists n_0 such that for $n \geq n_0$, the system H_n admits an orbit Γ_n at energy $\frac{1}{2}$ whose projection $\Pi(\Gamma_n)$ is δ -dense in $\Pi(H_n^{-1}(\frac{1}{2}))$, in the sense that the δ -neighborhood of $\Pi(\Gamma_n)$ in \mathbb{R}^3 covers $\Pi(H_n^{-1}(\frac{1}{2}))$.

1 Introduction and main result

The aim of this paper is to construct a simple class of *a priori* stable nearly integrable systems on \mathbb{A}^3 for which the dynamical behavior caused by a double resonance plays the central role and yields the existence of “asymptotically dense projected orbits”, that is, orbits at fixed energy whose projection on the energy level passes within an arbitrarily small distance from each point of the projected energy level, when the size of the perturbation tends to 0.

Several more general results on Arnold diffusion were recently announced. The goal of this paper is more modest, we try to underline the geometry of the diffusion process and

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to get rid of all (heavy) technical details as far as possible. We however think that our present work can help understand the more sophisticated methods coming into play for the proof of the so-called Arnold conjecture. We refer to the forthcoming publications by C.-Q. Cheng, V. Kaloshin and Ke Zhang, and J.-P. Marco for the complete studies (see [C12, Mar1, Mar2, KZ12]) .

Given an integer $m \geq 1$, we denote by $\mathbb{A}^m = \mathbb{T}^m \times \mathbb{R}^m$ the cotangent bundle of the torus \mathbb{T}^m that we endow with its usual angle-action coordinates (θ, r) and its Liouville symplectic form $\Omega = \sum_{i=1}^m dr_i \wedge d\theta_i$. We denote by Π the projection $\mathbb{A}^m \rightarrow \mathbb{R}^m$. When H is a C^κ function on an open set of \mathbb{A}^m , $\kappa \geq 2$, we denote by X^H its Hamiltonian vector field and by Φ_t^H its local flow. Given a function H and an element a in its range, we write $H^{-1}(a)$ instead of $H^{-1}(\{a\})$, even if H is not a bijection.

Our systems will be defined on \mathbb{A}^3 and have the following form

$$H_n(\theta, r) = \frac{1}{2}\|r\|^2 + \frac{1}{n}U(\theta_2, \theta_3) + f_n(\theta, r),$$

where $\|\cdot\|$ stands for the Euclidean norm, $U \in C^\kappa(\mathbb{T}^2)$ is a generic potential function and $f_n \in C^{\kappa-1}(\mathbb{A}^3)$ is an additional perturbation such that $\|f_n\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \frac{1}{n}$. For the sake of simplicity, we limit ourselves to the case where κ is an integer ≥ 2 but the construction could easily be extended to the C^∞ or Gevrey cases as well.

The system H_n is a perturbation of the integrable system $h(r) = \frac{1}{2}\|r\|^2$. We will focus on the energy level $H_n^{-1}(\frac{1}{2})$ but any other positive energy level would have the same properties. The frequency map associated with h is

$$\omega(r) = r,$$

and the double resonance under concern is the set of actions r such that $\omega_2(r) = \omega_3(r) = 0$, that is, the line $r_2 = r_3 = 0$. This line intersects the unperturbed level $\mathbb{S} = h^{-1}(\frac{1}{2})$ (the unit sphere) at the points $D_\pm = (\pm 1, 0, 0)$. Both averaged systems at these points have the same “principal part”, namely:

$$\overline{H}_n(\theta_2, \theta_3, r_2, r_3) = \frac{1}{2}(r_2^2 + r_3^2) + \frac{1}{n}U(\theta_2, \theta_3).$$

The full averaged systems also contain the average of f_n , but this will be insignificant thanks to a proper choice of this additional perturbation. The system \overline{H}_n is of “classical form”, the sum of a kinetic part and a potential part. The potential U is the main data of the problem, it will be arbitrarily chosen in a residual subset of $C^\kappa(\mathbb{T}^2)$. In particular, the system \overline{H}_n will be nonintegrable. This property is in contrast with the previous studies on double resonances where the averaged system was usually assumed to be integrable or nearly integrable (see [Bes97]). However, this nonintegrability and the associated “chaotic behavior” are essential features of generic nearly integrable systems as proved in the recent studies on Arnold diffusion. On the contrary, the last term f_n of the perturbation will be a “very nongeneric” bump function, especially designed to easily create and control the so-called “splitting of separatrices” in the spirit of [D88, MS02].

The truncated system

$$\mathcal{H}_n(\theta, r) = \frac{1}{2}\|r\|^2 + \frac{1}{n}U(\theta_2, \theta_3), \quad (\theta, r) \in \mathbb{A}^3, \quad (1)$$

does not admit diffusion orbits. In fact, it appears as the direct product of the one-degree-of-freedom Hamiltonian $\frac{1}{2}r_1^2$ with the previous system \overline{H}_n , and the conservation of energy in both factors prevents from any diffusion phenomenon. It is only when the perturbation f_n is added that the splitting of separatrices appears and makes the diffusion possible. The structure of our system is therefore in some sense analogous to that of Arnold's initial model for diffusion along a simple resonance ([A64]). But while in Arnold's model the diffusion phenomenon occurs only along a single resonance, in our model the diffusion takes place along a very large family of simple resonances, namely the great circles of \mathbb{S} orthogonal to the vectors $k = (0, k_2, k_3)$, where k_2, k_3 are coprime integers. The previous double resonant points D_{\pm} are the places where exchanges of resonances are made possible by the structure of the averaged systems in their neighborhood.

Let us now state our main result. For $2 \leq \kappa < +\infty$, we endow the spaces $C^\kappa(\mathbb{T}^2)$ of C^κ functions on \mathbb{T}^2 with their usual C^κ norms

$$\|U\|_{C^\kappa(\mathbb{T}^2)} = \max_{|\alpha| \leq \kappa} \max_{\theta \in \mathbb{T}^2} |\partial^\alpha U(\theta)|,$$

which make them Banach spaces. Throughout this paper, the triples $x = (x_1, x_2, x_3)$ in \mathbb{T}^3 or \mathbb{R}^3 will also be denoted by

$$x = (x_1, \overline{x}), \quad \overline{x} = (x_2, x_3).$$

We also introduce a formal definition for the notion of “approximative density”. Given a metric space (E, d) and $\delta > 0$, we say that a subset S of E is δ -dense in a subset $F \subset E$ when F is contained in the union of the family of all open δ -balls centered at points of S . We will prove the following diffusion result.

Theorem 1.1. *Let $\kappa \geq 2$ be a fixed integer. Then there exists a residual subset \mathcal{U} in $C^\kappa(\mathbb{T}^2)$ such that for each $U \in \mathcal{U}$, there exists a sequence $(f_n)_{n \geq 1}$ of $C^{\kappa-1}$ functions on \mathbb{A}^3 , with $\|f_n\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \frac{1}{n}$, such that for any $\delta > 0$, there exists n_0 such that for $n \geq n_0$, the system*

$$H_n(\theta, r) = \frac{1}{2}\|r\|^2 + \frac{1}{n}U(\overline{\theta}) + f_n(\theta, r), \quad (\theta, r) \in \mathbb{A}^3, \quad (2)$$

admits an orbit Γ_n with energy $\frac{1}{2}$ such that $\Pi(\Gamma_n)$ is δ -dense in $\Pi(H_n^{-1}(\frac{1}{2}))$.

Since we only aim at producing examples, the fact that \mathcal{U} is nonempty would be enough. The fact that the set of “convenient” potentials is residual was proved in [Mar1] but of course a single potential would be enough to construct an example. Since examples are not too difficult to produce, the content of this paper can be seen as independent from [Mar1]. However, taking for granted the residual character of \mathcal{U} makes it plausible –eventhough we do not try to prove this– that the wild behavior of orbits described in our examples is in fact “typical” for a priori stable perturbations of integrable systems. As mentionend above, this last question has been recently investigated by several authors.

We could also work in the class of diffeomorphisms, in which case an analogous construction yields examples of nearly-integrable diffeomorphisms with a large class of orbits biasymptotic to infinity. However we will limit ourselves here to the Hamiltonian case, which is indeed richer and slightly more difficult due to the additional geometrical difficulty induced by the preservation of energy. The paper is organized as follows.

- Section 2 is devoted to the description of those properties of classical systems which will be needed to construct our examples, namely the existence of suitable *chains of annuli*. Here we summarize [Mar1]. Again, we emphasize that our present construction is to a large extent independent of this latter work (apart from the necessary definitions), the concern of which is the genericity of the potential U for which the associated classical system possesses suitable chains of annuli.

- In Section 3, we deduce from the previous properties of classical systems the existence of *chains of cylinders* in our systems \mathcal{H}_n , and we prove that these chains project in the space of actions asymptotically close to a dense family of great circles in the unit sphere (the simple resonance lines). These cylinders are normally hyperbolic invariant manifolds diffeomorphic to $\mathbb{T}^2 \times [0, 1]$ and admit a foliation by invariant tori diffeomorphic to \mathbb{T}^2 .

- In Section 4, we construct the sequence (f_n) in such a way that each invariant torus in the previous family admits a homoclinic orbit along which its stable and unstable manifolds intersect transversely in a weak sense. This in particular yields the existence of heteroclinic connections between nearby enough tori contained in the same cylinders. Other transversality properties for heteroclinic orbits between tori belonging to distinct cylinders of the chains are also proved.

- Finally in Section 5, we prove the existence of the diffusion orbits. The key result there is the λ -lemma proved in [S13] which is specially designed for normally hyperbolic manifolds and which enables us to prove very easily the necessary shadowing results.

2 Classical systems

This section is devoted to the description of the generic hyperbolic properties of classical systems on the torus \mathbb{T}^2 which will be needed in the construction of our examples. Given a potential function $U \in C^\kappa(\mathbb{T}^2)$, we define here the associated classical system as the Hamiltonian on \mathbb{A}^2

$$C_U(x, y) = \frac{1}{2}\|y\|^2 + U(x), \quad (3)$$

where $x \in \mathbb{T}^2$ and $y \in \mathbb{R}^2$. We will always require the potential U to admit a single maximum \widehat{e} at some x^0 , which is nondegenerate in the sense that the Hessian of U is negative definite. This is of course true for a U in a residual subset $\mathcal{U}_0 \subset C^\kappa(\mathbb{T}^2)$. It is then easy to check that the lift of x^0 to the zero section of \mathbb{A}^2 is a hyperbolic fixed point for X^{C_U} .

1. We denote by $\pi : \mathbb{A}^2 \rightarrow \mathbb{T}^2$ the canonical projection and we fix $U \in \mathcal{U}_0$ together with the associated classical system $C := C_U$.

Definition 2.1. *Let $c \in H_1(\mathbb{T}^2, \mathbb{Z})$ and let $I \subset \mathbb{R}$ be an interval. An annulus for X^C realizing c and defined over I is a submanifold \mathbf{A} , contained in $C^{-1}(I) \subset \mathbb{A}^2$, such that*

- *for each $e \in I$, $\mathbf{A} \cap C^{-1}(e)$ is the orbit of a periodic solution γ_e of X^C , which is hyperbolic in $C^{-1}(e)$, with orientable stable and unstable bundles, and such that the projection $\pi \circ \gamma_e$ on \mathbb{T}^2 realizes c ;*
- *when $e > \bar{e}$, the frequency $\omega(e)$ of the solution γ_e is an increasing function of e and when $e < \bar{e}$, $\omega(e)$ is a decreasing function of e ;*

- *there exists a covering $I = \cup_{1 \leq i \leq i^*} I_i^*$ of I by open subintervals of I such that for $1 \leq i \leq i^*$ and for $e \in I_i^*$, the solution γ_e admits a homoclinic solution ω_e^i along which the stable and unstable manifolds of γ_e intersect transversely inside $C^{-1}(e)$.*
- *one can choose the covering $I = \cup_{1 \leq i \leq i^*} I_i^*$ in such a way that I_i and I_j are disjoint if $|j - i| \geq 2$ and the solutions ω_e^i and ω_e^{i+1} are geometrically disjoint for e in the intersection $I_i^* \cap I_{i+1}^*$.*

Since the solutions γ_e are hyperbolic and vary continuously with e (since A is assumed to be a submanifold), the annulus A is a $C^{\kappa-1}$ submanifold of \mathbb{A}^2 , with boundary $\partial A \sim \mathbb{T} \times \partial I$. It is clearly normally hyperbolic (the boundary causes no trouble in this simple setting, due to the conservation of the Hamiltonian), and its stable and unstable manifolds are the unions of those of the periodic solutions γ_e . Note that when I has a boundary point, the family γ_e can be continued over a slightly larger open interval, but it will be interesting to allow the intervals to be compact in our subsequent constructions.

It is not difficult to prove that there exists an embedding $\phi : \mathbb{T} \times I \rightarrow \mathbb{A}^2$ whose image is A and which satisfies

$$C \circ \phi(\varphi, e) = e. \quad (4)$$

Note that obviously $\phi(\mathbb{T} \times \{e\}) = A \cap C^{-1}(e)$. Moreover, one can find a symplectic embedding ϕ_s such that $C \circ \phi_s$ is in action-angle form (that is, does not depend on the angle), where of course $\mathbb{T} \times I$ is equipped with its usual symplectic structure.

2. Due to the reversibility of C , the solutions of the vector field X^C occur in *opposite pairs* (pairs of symmetric solutions whose time parametrizations are exchanged by the symmetry $t \mapsto -t$). This is in particular the case for the solutions homoclinic to the hyperbolic fixed point O associated with the maximum x^0 of U . We set

$$\widehat{e} = \text{Max } U = U(x^0).$$

Definition 2.2. *Let $c \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$. A singular annulus for X^C realizing $\pm c$, with parameters $\widehat{e} > \widehat{e}$ and $e^0 < \widehat{e}$, is a C^1 invariant manifold A_\bullet with boundary , diffeomorphic to the sphere S^2 minus three disjoint open discs, such that, setting $I =]\widehat{e}, \widehat{e}]$ and $I_0 = [e^0, \widehat{e}[:$*

- $A_\bullet \cap C^{-1}(\widehat{e})$ is the union of the hyperbolic fixed point O and a pair of opposite homoclinic orbits,
- $A_\bullet \cap C^{-1}(I)$ admits two connected components $A_\bullet^>$ and $A_\bullet^<$, which are annuli defined over I and realizing c and $-c$ respectively,
- $A_\bullet^0 = A_\bullet \cap C^{-1}(I_0)$ is an annulus defined over I_0 and realizing the null class 0,
- A_\bullet admits a C^1 stable (resp. unstable) manifold, in which the union of the stable (resp. unstable) manifolds of $A_\bullet^>$, $A_\bullet^<$ and A_\bullet^0 is dense.
- both homoclinic orbits admit homoclinic connections along which the stable and unstable manifolds of A_\bullet intersect transversely in $C^{-1}(\widehat{e})$.

Note that a singular annulus A_\bullet is “almost everywhere $C^{\kappa-1}$ ”, since the connected components of $A_\bullet \cap C^{-1}(I)$ and $A_\bullet \cap C^{-1}(I_0)$ are annuli, so $C^{\kappa-1}$ submanifolds of \mathbb{A}^2 . Note also that A_\bullet is a center manifold for both homoclinic orbits in $A_\bullet \cap C^{-1}(\hat{e})$, with hyperbolic transverse spectrum. One can in fact prove that a singular annulus is slightly more regular than C^1 (depending on the Lyapunov exponents of the fixed point O), but this is useless here.

A singular annulus is depicted in Figure 1: it is essentially the part of the phase space of a simple pendulum limited by two essential invariant curves at the same energy, from which a neighborhood of the elliptic fixed point was removed. More precisely, on the annulus A equipped with the coordinates (φ, I) , we define a “pendulum Hamiltonian” as a Hamiltonian of the form

$$P_{\hat{e}}(\varphi, I) = \frac{1}{2}I^2 + V(\varphi) + \hat{e}$$

where V is a C^2 potential function with a single nondegenerate maximum at 0 and a single nondegenerate minimum, which satisfies $V(0) = 0$. For $a < \hat{e} < b$ we introduce the subset $\mathcal{A}_\bullet(a, b)$ defined by $a \leq P_{\hat{e}}(\varphi, I) \leq b$. So $\mathcal{A}_\bullet(a, b)$ is the zone bounded by the two invariant curves of equation $P_{\hat{e}} = b$, together with an invariant curve surrounding the elliptic point. We call $\mathcal{A}_\bullet(\hat{e}, V, a, b)$ the *standard singular annulus* with parameters (\hat{e}, V, a, b) . One proves that a singular annulus is C^1 diffeomorphic to some standard annulus by a diffeomorphism $\phi_\bullet : \mathcal{A}_\bullet(a, b) \rightarrow A_\bullet$ such that

$$C|_{A_\bullet} \circ \phi_\bullet = P_{\hat{e}}. \quad (5)$$

Remark 2.3. By definition of an annulus, there exist embeddings $\phi_\bullet^> : \mathbb{T} \times]\hat{e}, \tilde{e}] \rightarrow A_\bullet^>$, $\phi_\bullet^< : \mathbb{T} \times]\hat{e}, \tilde{e}] \rightarrow A_\bullet^<$ and $\phi_\bullet^0 : \mathbb{T} \times [e^0, \hat{e}[\rightarrow A_\bullet^0$ for the 3 subannuli of a singular annulus. These embeddings satisfy

$$C \circ \phi_\bullet^*(\varphi, e) = e, \quad (6)$$

where $*$ stands for $>$, $<$ or 0 .

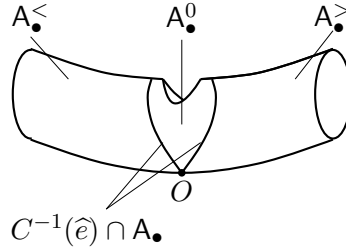


Figure 1: A singular annulus

3. Let us now turn to the definition of chains of annuli for the classical system C . We say that a family $(I_i)_{1 \leq i \leq m}$ of nontrivial closed subintervals of $]\hat{e}, +\infty[$ is *ordered* when $\text{Max } I_i > \text{Min } I_{i+1}$ for $1 \leq i \leq m - 1$. Of course in the following we will assume that $\text{Max } I_i$ is only slightly larger than $\text{Min } I_{i+1}$, so that the overlapping between two consecutive intervals is only a small neighborhood of their extremities.

Definition 2.4. Let $c, c' \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$.

- A chain of annuli realizing c is a family $(A_i)_{0 \leq i \leq m}$ of annuli realizing c , defined over an ordered family $(I_i)_{0 \leq i \leq m}$ of closed subintervals of $] \hat{e}, +\infty[$, with the additional property that for $0 \leq i \leq m-1$, if $\Gamma_i(e)$ stands for the hyperbolic orbit at energy e on A_i , if $e \in [\text{Min } I_{i+1}, \text{Max } I_i]$:

$$W^u(\Gamma_i(e)) \cap W^s(\Gamma_{i+1}(e)) \neq \emptyset, \quad W^s(\Gamma_i(e)) \cap W^u(\Gamma_{i+1}(e)) \neq \emptyset,$$

the intersection being transverse in the corresponding energy level $C^{-1}(e)$.

- A generalized chain of annuli realizing c and c' is the union of two chains $(A_i)_{0 \leq i \leq m}$ and $(A'_i)_{0 \leq i \leq m'}$ realizing c and c' respectively, together with a singular annulus A_\bullet , such that

$$W^u(A_0) \cap W^s(A_\bullet) \neq \emptyset, \quad W^s(A_0) \cap W^u(A_\bullet) \neq \emptyset,$$

$$W^u(A'_0) \cap W^s(A_\bullet) \neq \emptyset, \quad W^s(A'_0) \cap W^u(A_\bullet) \neq \emptyset,$$

the intersections being transverse in \mathbb{A}^2 .

Note that we do not specify the homology of the singular annulus A_\bullet , this latter turns out to be fixed independently of the classes c and c' in our subsequent construction.

4. We now state the genericity result from [Mar1]. We say that $c \in H_1(\mathbb{T}^2, \mathbb{Z}) \setminus \{0\}$ is primitive when the equality $c = mc'$ with $m \in \mathbb{Z}$ implies $m = \pm 1$. We denote by $\mathbf{H}_1(\mathbb{T}^2, \mathbb{Z})$ the set of primitive homology classes. Let \mathbf{d} be the Hausdorff distance for compact subsets of \mathbb{R}^2 and $\Pi : \mathbb{A}^2 \rightarrow \mathbb{R}^2$ the canonical projection. Recall that \mathcal{U}_0 is the set of potentials with a single nondegenerate maximum.

Theorem 2.5. ([Mar1]). *Let $2 \leq \kappa \leq +\infty$. Then there exists a residual subset $\mathcal{U} \subset \mathcal{U}_0$ in $C^\kappa(\mathbb{T}^2)$ such that for $U \in \mathcal{U}$, the associated classical system C_U defined in (3) satisfies the following properties.*

1. For each $c \in \mathbf{H}_1(\mathbb{T}^2, \mathbb{Z})$ there exists a chain $\mathbf{A}(c) = (A_0, \dots, A_m)$ of annuli realizing c , defined over ordered intervals I_0, \dots, I_m , such that the first and last intervals are of the form

$$I_0 =] \text{Max } U, e_m] \quad \text{and} \quad I_m = [e_P, +\infty[,$$

for suitable constants e_m and e_P .

2. Let $c = (c_1, c_2)$ be the canonical identification of $H_1(\mathbb{T}^2, \mathbb{Z})$ with \mathbb{Z}^2 and for $e > 0$, set

$$Y_c(e) = \frac{\sqrt{2e} c}{\|c\|}$$

Setting $\Gamma_e = A_m \cap C_U^{-1}(e)$ for $e \in [e_P, +\infty[$, then

$$\lim_{e \rightarrow +\infty} \mathbf{d}(\Pi(\Gamma_e), \{Y_c(e)\}) = 0$$

3. There exists a singular annulus A_\bullet which admits transverse heteroclinic connections with every first annulus in the previous chains.

The existence of the “high energy annuli” A_m is proved by a simple argument *à la* Poincaré, on the creation of hyperbolic orbits near perturbations of resonant tori, so we call e_P the Poincaré energy for the class c . The other annuli are proved to exist by minimization arguments of Morse and Hedlund.

There exist in general several singular annuli with the previous intersection property, but one will be enough for our future needs. We say that a chain with I_0 and I_m as in the first item above is *biasymptotic to $\hat{e} := \text{Max } U$ and $+\infty$* . It may be useful to rephrase Theorem 2.5 in a more concise way.

Corollary 2.6. *For $U \in \mathcal{U}$ and for each pair of classes $c, c' \in \mathbf{H}_1(\mathbb{T}^2, \mathbb{Z})$, there exists a generalized chain of annuli, union of $(A_i)_{0 \leq i \leq m}$, $(A'_i)_{0 \leq i \leq m'}$ and A_\bullet , such that $(A_i)_{0 \leq i \leq m}$ and $(A'_i)_{0 \leq i \leq m'}$ are biasymptotic to \hat{e} and $+\infty$ and realize c and c' respectively.*

In the y -plane, one therefore gets the following picture for the projection of generalized chains of annuli, along some lines of rational slope (which obviously correspond to integer homology classes when the energy tends to $+\infty$, by Theorem 2.5, 2).

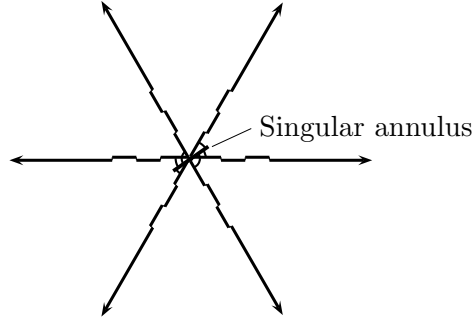


Figure 2: Projected generalized chains of annuli in a classical system

3 Chains of cylinders for \mathcal{H}_n

Here we call *cylinder* for a vector field defined on the cylinder \mathbb{A}^3 a normally hyperbolic invariant manifold with boundary, diffeomorphic to $\mathbb{T}^2 \times [0, 1]$. In particular, the stable and unstable manifolds of a cylinder are well-defined and the definition of heteroclinic connections between cylinders can be properly stated. In this section, we prove the existence of a family of chains of cylinders for the truncated system \mathcal{H}_n defined in (1), in the energy level $\mathcal{H}_n^{-1}(\frac{1}{2})$, whose projection by Π forms an asymptotically dense subset of the unit sphere.

3.1 Cylinders and chains

1. Let us set out a first definition in a context adapted to our needs (but which can obviously be adapted to more general ones).

Definition 3.1. *Let X be a vector field on \mathbb{A}^3 .*

- We say that $\mathcal{C} \subset \mathbb{A}^3$ is a C^p invariant cylinder with boundary for X if \mathcal{C} is a submanifold of \mathbb{A}^3 , C^p -diffeomorphic to $\mathbb{T}^2 \times [0, 1]$, such that X is everywhere tangent to \mathcal{C} and is moreover tangent to $\partial\mathcal{C}$ at each point of $\partial\mathcal{C}$.
- Given an invariant cylinder with boundary \mathcal{C} , we say that it is normally hyperbolic when there exist a neighborhood N of \mathcal{C} and a complete vector field X_\circ on \mathbb{A}^3 such that $X \equiv X_\circ$ in N and such that X_\circ admits a normally hyperbolic invariant submanifold \mathcal{C}_\circ , diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, which contains \mathcal{C} .

Note first that \mathcal{C} is invariant under the flow, thanks to the tangency hypothesis on the boundary. In particular, both connected components of $\partial\mathcal{C}$ are invariant 2-dimensional tori. In the following, when the context is clear, normally hyperbolic cylinders with boundary will be called *compact invariant cylinders* for short.

The main interest of the previous definition is that it is possible to properly define the stable and unstable manifolds of compact invariant cylinders. Indeed, one checks that the stable manifold $W^{ss}(x)$ of a point $x \in \mathcal{C}$ is well-defined and *independent of the choice of $(X_\circ, \mathcal{C}_\circ)$* (recall that $W^{ss}(x)$ is the set of all initial conditions y such that $\text{dist}(\Phi^{tX}(x), \Phi^{tX}(y))$ tends to 0 at an exponential rate e^{-ct} , where c dominates the contraction exponent on \mathcal{C}_\circ). The stable manifold of \mathcal{C} is then well-defined as the union of the stable manifolds $W^{ss}(x)$ for $x \in \mathcal{C}$, which turns out to have the same regularity as \mathcal{C} . The same remark obviously also holds for the unstable manifolds.

2. In addition to our previous invariant cylinders, it will be necessary to introduce slightly more complicated objects which we call *singular cylinders*. Recall that \mathcal{A}_\bullet is the standard singular annulus defined in the previous section.

Definition 3.2. Let X be a vector field on \mathbb{A}^3 .

- We say that $\mathcal{C}_\bullet \subset \mathbb{A}^3$ is an invariant singular cylinder for X if \mathcal{C}_\bullet is a C^1 submanifold with boundary of \mathbb{A}^3 , diffeomorphic to $\mathbb{T} \times \mathcal{A}_\bullet$, such that X is everywhere tangent to \mathcal{C}_\bullet (and is moreover tangent to $\partial\mathcal{C}_\bullet$ at each point of $\partial\mathcal{C}_\bullet$).
- Given an invariant singular cylinder \mathcal{C}_\bullet , we say that it is normally hyperbolic when there exist a neighborhood N of \mathcal{C}_\bullet and a complete vector field X_\circ on \mathbb{A}^3 such that $X \equiv X_\circ$ on N and which admits a normally hyperbolic invariant submanifold \mathcal{C}_\circ , diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}$, which contains \mathcal{C}_\bullet .

As above, we simply say *compact singular cylinders* instead of normally hyperbolic compact invariant singular cylinders. Again, the stable and unstable manifolds of a point $x \in \mathcal{C}_\bullet$ are well-defined and independent of the choice of $(X_\circ, \mathcal{C}_\circ)$, and this is also the case for the stable and unstable manifolds of \mathcal{C}_\bullet .

3. Let H be a Hamiltonian on \mathbb{A}^3 and let \mathbf{e} be a regular value of H .

Definition 3.3. A chain of cylinders for H at energy \mathbf{e} is a finite family $(\mathcal{C}_i)_{1 \leq i \leq i^*}$ of compact invariant cylinders or singular cylinders, contained in $H^{-1}(\mathbf{e})$, such that $W^u(\mathcal{C}_i)$ intersects $W^s(\mathcal{C}_{i+1})$ for $1 \leq i \leq i^* - 1$.

Note in particular that, for the sake of simplicity, we do not make any distinction between “regular” cylinders and singular cylinders in a chain. Note also that the definition

here slightly differs from that of chains of annuli above. In the following we will have to add suitable transversality conditions for the various homoclinic and heteroclinic intersections in a chain of cylinders, which could be stated in a general context but will be easier to make explicit in the case of our Hamiltonians H_n , this will be done in Section 4.

3.2 Cylinders for \mathcal{H}_n

1. We now go back to the truncated Hamiltonian \mathcal{H}_n defined in (1). Let $k = (k_2, k_3) \in \mathbb{Z}^2$ be a primitive integer vector (that is, k_2 and k_3 are coprime) and let \mathbb{S}_k be the half great circle of the unit sphere \mathbb{S} formed by the actions $r = (r_1, \bar{r}) = (r_1, r_2, r_3)$ such that

$$\bar{r} \cdot k = 0, \quad (-r_3, r_2) \cdot k \geq 0, \quad r \in \mathbb{S}.$$

We denote by \mathbf{d} the Hausdorff distance for compact subsets of \mathbb{R}^3 . The main result of this section is the following.

Proposition 3.4. *Let $U \in \mathcal{U}$ and set, for $(\theta, r) \in \mathbb{A}^3$*

$$\mathcal{H}_n(\theta, r) = \frac{1}{2}\|r\|^2 + \frac{1}{n}U(\theta_2, \theta_3).$$

Fix k as above and fix $\delta > 0$. Then:

- *there is $n_0(k) > 0$ such that for $n \geq n_0(k)$, there are regular cylinders $\mathcal{C}_{-m}, \dots, \mathcal{C}_{-1}, \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_m$, where the integer m depends on k , which satisfy*

$$\mathbf{d}(\cup_j \Pi(\mathcal{C}_j), \mathbb{S}_k) < \delta, \quad (7)$$

and such that both ordered families $\mathcal{C}_{-m}, \dots, \mathcal{C}_{-1}, \mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_m$ and $\mathcal{C}_m, \dots, \mathcal{C}_1, \mathcal{C}_0, \mathcal{C}_{-1}, \dots, \mathcal{C}_{-m}$ are chains;

- *there exist two singular cylinders \mathcal{C}_\bullet^- and \mathcal{C}_\bullet^+ , independent of k , such that the extremal cylinders \mathcal{C}_{-m} and \mathcal{C}_m admit transverse heteroclinic connections with \mathcal{C}_\bullet^- and \mathcal{C}_\bullet^+ respectively;*
- *each cylinder \mathcal{C}_j admits a foliation by isotropic \mathbb{T}^2 tori, such that the union of the subfamily of dynamically minimal tori is a dense subset of \mathcal{C}_j , and each singular cylinder \mathcal{C}_\bullet^\pm admits a foliation by isotropic \mathbb{T}^2 tori on an open and dense subset, such that the union of the subfamily of dynamically minimal tori is a dense subset of \mathcal{C}_\bullet^\pm .*

We will moreover prove that the cylinders \mathcal{C}_j and \mathcal{C}_{-j} are exchanged with one another by a natural symmetry.

Proof. We can assume without loss of generality that $\hat{e} = \text{Max } U = 0$. We first perform a standard rescaling to get rid of the parameter n , namely, setting for $a > 0$:

$$\sigma_a(\theta, r) = (\theta, ar), \quad (8)$$

one immediately checks the conjugacy relation

$$\Phi_{nt}^{\mathcal{H}_n} = \sigma_{\sqrt{n}}^{-1} \circ \Phi_t^{\mathcal{H}} \circ \sigma_{\sqrt{n}}, \quad (9)$$

where $\mathcal{H} := \mathcal{H}_1$, while $\sigma_{\sqrt{n}}$ sends the energy level $\mathcal{H}_n^{-1}(\frac{1}{2})$ onto the level $\mathcal{H}^{-1}(\frac{n}{2})$. We can therefore examine the behavior of the system \mathcal{H} at high energy \mathbf{e} and get our results by inverse rescaling. We will fix two coprime integers (k_2, k_3) and concentrate on the neighborhood of the half great circle $\sqrt{2\mathbf{e}}\mathbb{S}_k$ on the sphere of radius $\sqrt{2\mathbf{e}}$.

- We will apply Theorem 2.5 to $c \sim (-k_3, k_2)$. Reversing the order of the intervals in this theorem, for compatibility reasons with the statement of Proposition 3.4, there is an ordered family I_m, \dots, I_0 , with $I_m =]0, e_m]$ and $I_0 = [e_P, +\infty[$ such that the system C_U admits a chain of annuli $\mathbf{A}_m, \dots, \mathbf{A}_0$ realizing c and defined over I_m, \dots, I_0 . For $0 \leq j \leq m$, we denote by $\phi_j : \mathbb{T} \times I_j \rightarrow \mathbb{A}^2$ the embedding of \mathbf{A}_j satisfying (4), and for $e \in I_j$, we denote by $\Gamma_j(e) = \phi_j(\mathbb{T} \times \{e\})$ the periodic orbit at energy e in \mathbf{A}_j .

- Let us fix an energy $\mathbf{e} > e_P$. The level $\mathcal{H}^{-1}(\mathbf{e})$ contains the union

$$\bigcup_{0 \leq e_1 \leq \mathbf{e}} \left\{ \theta_1 \in \mathbb{T}, \frac{1}{2}r_1^2 = e_1 \right\} \times C_U^{-1}(\mathbf{e} - e_1).$$

which will serve as a guide to construct embeddings for our cylinders.

- Consider first an annulus \mathbf{A}_j with $1 \leq j \leq m$ and set $I_j = [a_j, b_j]$ if $1 \leq j \leq m-1$ and $I_m =]0, b_m]$, so that I_j is contained in $]0, e_P]$ (recall that $I_0 = [e_P, +\infty[$). We introduce the map

$$\begin{aligned} F_j^+ : \mathbb{T}^2 \times I_j &\longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \subset \mathbb{A} \times \mathbb{A}^2 \\ F_j^+(\varphi_1, \varphi_2, e) &= \left((\varphi_1, \sqrt{2(\mathbf{e} - e)}), \phi_j(\varphi_2, e) \right). \end{aligned} \quad (10)$$

One immediately checks that F_j^+ is an embedding. Let $\mathcal{C}_j \subset \mathcal{H}^{-1}(\mathbf{e})$ be its image. Then \mathcal{C}_j is a cylinder (diffeomorphic to the product of \mathbb{T}^2 with an interval), which is a circle bundle over the annulus \mathbf{A}_j . So \mathcal{C}_j admits a regular foliation whose leaves are the tori

$$\mathcal{T}_e = F_j^+(\mathbb{T}^2 \times \{e\}).$$

The torus \mathcal{T}_e is the direct product of the circle $\mathbb{T} \times \{\sqrt{2(\mathbf{e} - e)}\}$ in the first factor of the product $\mathbb{A} \times \mathbb{A}^2$ with the hyperbolic periodic orbit $\Gamma_j(e)$ in the second factor. For each point $z = (z_1, z_2) \in \mathbb{A} \times \mathbb{A}^2$ in \mathcal{C}_j , there is a single hyperbolic orbit in the annulus \mathbf{A}_j which contains z_2 . This yields a decomposition of the tangent space $T_z \mathcal{H}^{-1}(\mathbf{e})$ as a sum $T_z \mathcal{C}_j \oplus E^+(z) \oplus E^-(z)$, where $E^\pm(z)$ are the stable and unstable directions of that hyperbolic orbit at the point z_2 . All these considerations also make sense for any small enough hyperbolic continuation of \mathbf{A}_j , which immediately proves that \mathcal{C}_j is a compact invariant cylinder in the sense of Definition 3.1.

- One gets a parallel construction using the embedding

$$\begin{aligned} F_j^- : \mathbb{T}^2 \times I_j &\longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \subset \mathbb{A} \times \mathbb{A}^2 \\ F_j^-(\varphi_1, \varphi_2, e) &= \left((\varphi_1, -\sqrt{2(\mathbf{e} - e)}), \phi_j(\varphi_2, e) \right). \end{aligned} \quad (11)$$

whose image will be denoted by \mathcal{C}_{-j} and is a compact invariant cylinder as well. Moreover, \mathcal{C}_j and \mathcal{C}_{-j} are obviously symmetric under $r_1 \mapsto -r_1$.

- To exhibit the singular cylinders, one fixes an embedding $\phi_\bullet : \mathcal{A}_\bullet \rightarrow \mathbb{A}^2$ satisfying (5) (with suitable parameters), whose image is the singular annulus \mathbf{A}_\bullet of the system C_U

depicted in Theorem 2.5. This enables one to introduce two maps

$$\begin{aligned} F_{\bullet}^{\pm} : \mathbb{T} \times \mathcal{A}_{\bullet} &\longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \\ F_{\bullet}^{\pm}(\varphi_1, (\varphi_2, r_2)) &= \left(\left(\varphi_1, \pm \sqrt{2(\mathbf{e} - C_U(\phi_{\bullet}(\varphi_2, r_2)))} \right), \phi_{\bullet}(\varphi_2, r_2) \right). \end{aligned} \quad (12)$$

Again, one easily checks that these are embeddings. Their images $\mathcal{C}_{\bullet}^{\pm}$ are singular cylinders for \mathcal{H} , which are symmetric under $r_1 \mapsto -r_1$.

• The case of \mathbf{A}_0 is slightly different, since the energy e can no longer be used as a global parameter on it. Rather, we introduce the interval $J_0 = [-\sqrt{2(\mathbf{e} - e_P)}, \sqrt{2(\mathbf{e} - e_P)}]$ and the map

$$\begin{aligned} F_0 : \mathbb{T}^2 \times J_0 &\longrightarrow \mathcal{H}^{-1}(\mathbf{e}) \subset \mathbb{A} \times \mathbb{A}^2 \\ F_0(\varphi_1, \varphi_2, r_1) &= \left((\varphi_1, r_1), \phi_0(\varphi_2, \mathbf{e} - \tfrac{1}{2}r_1^2) \right). \end{aligned} \quad (13)$$

One easily checks that F_0 is again an embedding and that its image \mathcal{C}_0 is a compact invariant cylinder (note that now \mathcal{C}_0 is a two sheeted ramified circle bundle over the corresponding part of \mathbf{A}_0). This completes the construction of the family $\mathcal{C}_{-m}, \dots, \mathcal{C}_m$ and $\mathcal{C}_{\bullet}^{\pm}$.

• Fix now an integer $j \in \{0, \dots, m-1\}$ and fix $e \in [\text{Max } I_{j+1}, \text{Min } I_j]$. By definition of a chain of annuli, there exists a heteroclinic connection

$$\Omega_j^{j+1} \subset W^u(\Gamma_j(e)) \cap W^s(\Gamma_{j+1}(e))$$

between extremal periodic orbits of \mathbf{A}_j and \mathbf{A}_{j+1} , which gives rise to an *annulus* of heteroclinic orbits (diffeomorphic to \mathbb{A}) between \mathcal{C}_j and \mathcal{C}_{j+1} , namely

$$(\mathbb{T} \times \{\sqrt{2(\mathbf{e} - e)}\}) \times \Omega_j^{j+1}.$$

Again, a parallel construction using now the heteroclinic connection

$$\Omega_{j+1}^j \subset W^u(\Gamma_{j+1}(e)) \cap W^s(\Gamma_j(e))$$

proves the existence of an annulus of heteroclinic orbits between $\mathcal{C}_{-(j+1)}$ and \mathcal{C}_{-j} . We indeed get more heteroclinic connections by considering the whole overlapping energy interval $[\text{Max } I_{j+1}, \text{Min } I_j]$. This proves that the family $\mathcal{C}_{-m}, \dots, \mathcal{C}_m$ is a chain of cylinders.

The proof for the opposite ordering $\mathcal{C}_m, \dots, \mathcal{C}_{-m}$ is similar. Finally, the existence of annuli of heteroclinic connections between $\mathcal{C}_{\pm m}$ and $\mathcal{C}_{\bullet}^{\pm}$ follows from exactly the same considerations as above.

• The cylinders \mathcal{C}_j , $1 \leq j \leq m$, are foliated by the invariant tori \mathcal{T}_e for $e \in I_j$. Let us prove that, when \mathbf{e} is large enough, they are dynamically minimal for e in a dense subset of I_j . Fix the integer j and let $\omega_j : I_j \rightarrow \mathbb{R}$ be the frequency map of the annulus \mathbf{A}_j . The frequency on the first factor (according to the component form of F_j^+) of \mathcal{T}_e , that is, the circle $\mathbb{T} \times \{\sqrt{\mathbf{e} - e}\}$ is $\sqrt{\mathbf{e} - e}$, so that the frequency vector of \mathcal{T}_e is

$$\Omega(e) = (\sqrt{\mathbf{e} - e}, \omega_j(e)).$$

Now, by Definition 2.1 and by compactness of I_j , the frequency map $\omega_j : I_j \rightarrow \mathbb{R}$ of the annulus \mathbf{A}_j satisfies

$$\omega_j'(e) \geq \mu > 0$$

for $e \in I_j$. This proves that the frequency curve $\Omega(e) \subset \mathbb{R}^2$ of \mathcal{C}_j is transverse to each vector line in \mathbb{R}^2 , so that the ratio Ω_2/Ω_1 is irrational for e in a dense subset of I_j . The corresponding torus \mathcal{T}_e is therefore dynamically minimal. Similar arguments show the same property for the cylinders \mathcal{C}_j , $-m \leq j \leq -1$, as well as for the singular cylinders \mathcal{C}_\bullet^\pm .

- It remains to examine the torsion properties of \mathcal{C}_0 . Observe that up to a standard linear change of variables (see [Mar1]), one can assume that $c = (1, 0)$. In the new variables, that we still denote by (θ, r) , the kinetic part of the Hamiltonian \mathcal{H} takes the form

$$T(r) = \frac{1}{2}(r_1^2 + Q(r_2, r_3)),$$

and for \mathbf{e} large enough X^T is the dominant term of $X^{\mathcal{H}}$ since X^U is bounded. Moreover, when the energy e of the classical system is large enough,

$$C_u(\theta_2, \theta_3, r_2, r_3) \sim \frac{1}{2}Q(r_2, r_3).$$

Now, by the asymptotic property of the projection of the Poincaré annulus \mathbf{A}_0 (that is, the annulus \mathbf{A}_m in Theorem 2.5), the frequency map of \mathbf{A}_0 is a $o(1)$ C^2 perturbation of the map

$$\omega : e \sim \frac{1}{2}Q(r_2, r_3) \mapsto \partial_{r_2}Q(r_2, r_3).$$

As a consequence the frequency vector of the corresponding tori $\pm\mathcal{T}(e)$ on \mathcal{C}_0 is a small C^2 perturbation of

$$\Omega(e) = (\sqrt{\mathbf{e} - e}, \omega(e)).$$

Since $\omega'(e) \rightarrow +\infty$ when $e \rightarrow \infty$, this proves now the density of the torsion property for the two connected components of \mathcal{C}_0 fibered over the subannulus of \mathbf{A}_0 defined over $[e^*, \mathbf{e}]$, where e^* is large enough (we obviously always require $\mathbf{e} > e^*$ for being consistent).

The case of the remaining components of \mathcal{C}_0 associated with the subannulus defined over $[e_P, e^*]$ is analogous to that of the cylinders \mathcal{C}_j since one can choose on this part a parametrization similar to those given by the embeddings F_j^\pm .

As a consequence, the union of the subset of dynamically minimal tori of \mathcal{C}_0 is dense in \mathcal{C}_0 .

- It only remains to prove (7), but this is an immediate consequence of Theorem 2.5, taking into account the reparametrization (9). This concludes the proof. \square

Definition 3.5. *We will say that the cylinders $\mathcal{C}_{\pm j}$ exhibited in Proposition 3.4 are associated with the annulus \mathbf{A}_j , for $1 \leq j \leq m$, and that \mathcal{C}_\bullet^\pm is associated with \mathbf{A}_\bullet . In the same way, we say that the embeddings F_j^\pm and F_\bullet^\pm are associated with the cylinders $\mathcal{C}_{\pm j}$ and \mathcal{C}_\bullet^\pm . Note that each (singular) embedding F_\bullet^+ or F_\bullet^- gives rise to three regular embeddings associated with the embeddings $\phi_\bullet^>$, $\phi_\bullet^<$ and ϕ_\bullet^0 of Remark 2.3.*

We will in general denote by $\mathbb{T}^2 \times \mathbb{I}$ the domain of an embedding associated with a regular cylinder, without specifying the particular form of the parametrization, and we will get rid of the various indices for the embeddings.

In the following we will apply the previous proposition to an increasing family of simple resonances $\cup_{1 \leq \ell \leq n} \mathbb{S}_{k_\ell}$, and we need to exhibit a single chain of cylinders for \mathcal{H}_n

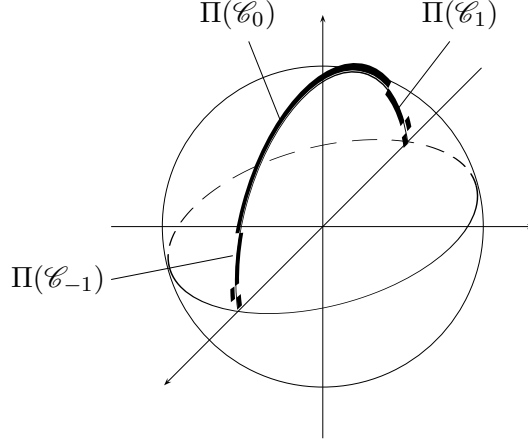


Figure 3: Projected cylinders

whose projection follows each semi-circle in this family. To this aim, we order the subset $\widehat{\mathbb{Z}}^2 \subset \mathbb{Z}^2$ formed by the primitive vectors k , in such a way that the resulting sequence $(k_\ell)_{\ell \geq 1}$ satisfies $\|k_\ell\| \leq \|k_{\ell+1}\|$. For each $k \sim c \in \widehat{\mathbb{Z}}^2$, we denote by $\text{Cyl}_k(\mathcal{H}_n)$ the set of cylinders associated with the annuli $A_0(c), \dots, A_m(c)$ of Theorem 2.5, together with the singular cylinders \mathcal{C}_\bullet^\pm , so that $\#\text{Cyl}_k(\mathcal{H}_n) = 2m + 3$. Finally we set

$$\text{Cyl}(\mathcal{H}_n) = \bigcup_{1 \leq \ell \leq n} \text{Cyl}_{k_\ell}(\mathcal{H}_n).$$

Recall that by Proposition 3.4, for each $k \in \widehat{\mathbb{Z}}^2$, the cylinders in $\text{Cyl}_k(\mathcal{H}_n)$ form *two* chains depending on the way they are ordered, namely:

$$\text{Chain}_k^+(\mathcal{H}_n) : \quad \mathcal{C}_\bullet^-, \mathcal{C}_{-m}, \dots, \mathcal{C}_0, \dots, \mathcal{C}_m, \mathcal{C}_\bullet^+,$$

and

$$\text{Chain}_k^-(\mathcal{H}_n) : \quad \mathcal{C}_\bullet^+, \mathcal{C}_m, \dots, \mathcal{C}_0, \dots, \mathcal{C}_{-m}, \mathcal{C}_\bullet^-.$$

Definition 3.6. Let $n \geq 1$ be fixed. We denote by $\text{Chain}(\mathcal{H}_n)$ the chain formed by the concatenation of the chains $\text{Chain}_{k_\ell}^{(-1)^\ell}(\mathcal{H}_n)$, $1 \leq \ell \leq n$.

Finally, we denote by $\text{Tori}(\mathcal{H}_n)$ the set of all \mathbb{T}^2 -tori of the form \mathcal{T}_e contained in the cylinders and singular cylinders of $\text{Cyl}(\mathcal{H}_n)$. The rest of the paper is devoted to the construction of a perturbation which will create shadowing orbits along $\text{Chain}(\mathcal{H}_n)$, passing close to a δ -dense family of dynamically minimal tori in $\text{Tori}(\mathcal{H}_n)$.

4 Construction of the perturbation

In this section, we will construct the perturbation $f_n \in C^{\kappa-1}(\mathbb{A}^3)$ such that the system $H_n = \mathcal{H}_n + f_n$ will admit the same family of cylinders as \mathcal{H}_n , with additional transversality properties for their invariant manifolds (the so-called splitting of separatrices).

4.1 The transversality conditions

We first set out some definitions for the splitting of separatrices. In this section f_n denotes a function in $C^{\kappa-1}(\mathbb{A}^3)$ whose support is contained in the complement of the union of the cylinders of $\text{Cyl}(\mathcal{H}_n)$. As a consequence, each $\mathcal{C} \in \text{Cyl}(\mathcal{H}_n)$ is also invariant under the flow generated by $H_n = \mathcal{H}_n + f_n$ and contained in $H_n^{-1}(\frac{1}{2})$. We can therefore set

$$\text{Cyl}(H_n) := \text{Cyl}(\mathcal{H}_n), \quad \text{Chain}(H_n) := \text{Chain}(\mathcal{H}_n), \quad \text{Tori}(H_n) := \text{Tori}(\mathcal{H}_n).$$

Given a point x in a cylinder of $\text{Cyl}(H_n)$, note that its stable and unstable manifolds $W^{s,u}(x)$ are well-defined, and that this is also the case for the stable and unstable manifolds of any $\mathcal{T} \in \text{Tori}(H_n)$, that we denote by $W^{s,u}(\mathcal{T})$. Let us now introduce our conditions.

Definition 4.1. Let $\mathcal{T} \in \mathcal{C}$ and $\mathcal{T}' \in \mathcal{C}'$ be two elements of $\text{Tori}(H_n)$ (recall that \mathcal{C} and \mathcal{C}' can be singular cylinders). We say that the pair $(\mathcal{T}, \mathcal{T}')$ satisfies the weak splitting condition if there exists $a \in \mathcal{T}$ whose unstable manifold $W^{uu}(a)$ intersects $W^s(\mathcal{C}')$ transversely in $H_n^{-1}(\frac{1}{2})$ at some point of $W^s(\mathcal{T}')$.

Note that our definition differs from the usual ones in which one requires the Lagrangian invariant manifolds of \mathcal{T} and \mathcal{T}' to intersect transversely. This latter splitting condition is obviously stronger than ours.

Definition 4.2. Let \mathcal{C} be a regular cylinder in $\text{Cyl}(H_n)$ and let F be the associated embedding, $F : \mathbb{T}^2 \times \mathbb{I} \rightarrow H_n^{-1}(\frac{1}{2})$ and for $s \in \mathbb{I}$, set $\mathcal{T}(s) = F(\mathbb{T}^2 \times \{s\})$. We say that \mathcal{C} satisfies Condition (T) if there exists $\rho > 0$ such that for each pair $(s, s') \in \mathbb{I}^2$ with $|s - s'| < \rho$, the pair $(\mathcal{T}(s), \mathcal{T}(s'))$ satisfies the weak splitting condition.

Let now $\mathcal{C}_\bullet = \mathcal{C}_\bullet^\pm$ be a singular cylinder, together with its subcylinders \mathcal{C}_\bullet^* , where $*$ stands for $>, <$ or 0 , and the associated embeddings F_\bullet^* defined over \mathbb{I}^* . Note that, by the construction of the previous section, the natural parameter in \mathbb{I}^* is the energy e of the classical system C_U . We set $\mathcal{T}^*(e) = F_\bullet^*(\mathbb{T}^2 \times \{e\})$.

Definition 4.3. We say that \mathcal{C}_\bullet satisfies Condition (T) when each subcylinder \mathcal{C}_\bullet^* satisfies Condition (T) and when moreover there exists $\rho > 0$ such that if $|e - e'| < \rho$, each pair $(\mathcal{T}^<(e), \mathcal{T}^0(e'))$ and $(\mathcal{T}^>(e), \mathcal{T}^0(e'))$ satisfies the weak splitting condition.

In other words, we want two close enough subtori in \mathcal{C}_\bullet to satisfy the weak splitting condition, when one of them is located in \mathcal{C}_\bullet^0 and the other one is in $\mathcal{C}_\bullet^>$ or $\mathcal{C}_\bullet^<$.

As for chains now, we have to add a transversality condition for tori contained in distinct cylinders.

Definition 4.4. We say that a chain of cylinders $(\mathcal{C}_k)_{1 \leq k \leq k^*}$ satisfies Condition (S) when each cylinder \mathcal{C}_k satisfies Condition (T) and when moreover, for $1 \leq k \leq k^* - 1$, there are open subsets $O_k \subset \mathcal{C}_k$ and $O_{k+1} \subset \mathcal{C}_{k+1}$, union of elements of $\text{Tori}(H_n)$, such that for each $\mathcal{T} \subset O_k$ and $\mathcal{T}' \in O_{k+1}$, the pair $(\mathcal{T}, \mathcal{T}')$ satisfies the weak splitting condition.

Note that Condition (S) is obviously open in the following sense.

Lemma 4.5. Assume that $\text{Chain}(H_n)$ satisfies Condition (S). Given a small enough function f in the C^2 topology, with support contained in the complement of the union of the cylinders of $\text{Cyl}(H_n)$, then $\text{Chain}(H_n + f) := \text{Chain}(H_n)$ is a chain at energy $\frac{1}{2}$ for $H_n + f$ and satisfies Condition (S).

Our purpose in this section is to prove the following result.

Proposition 4.6. *Fix $\kappa \geq 2$. Then for each $n \geq 1$, there exists a function $f_n \in C^{\kappa-1}(\mathbb{A}^3)$, whose support is contained in the complement of the union of the cylinders of $\text{Cyl}(\mathcal{H}_n)$, and which satisfies $\|f_n\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \frac{1}{n}$, such that $\text{Chain}(\mathcal{H}_n + f_n)$ satisfies Condition (S).*

The rest of the section is devoted to the proof which requires two steps. We will first exhibit a perturbation which creates the heteroclinic connections for the pairs of tori contained in the same cylinder and we will then use the previous openness property to add a second perturbation adapted to the heteroclinic conditions for extremal tori of the chain.

4.2 Flow boxes near homoclinic intersections of cylinders

In order to properly define the various perturbations, we first need to construct “flow boxes” centered on suitable parts of the homoclinic manifolds of the cylinders of \mathcal{H}_n , and located “far from these cylinders”. Given $U \in \mathcal{U}$, we fix an annulus A of the system C_U defined over I . We denote by $\Gamma(e)$ the periodic orbit $A \cap C_U^{-1}(e)$ and we fix an open subinterval $I^* \subset I$ over which $W^{s,u}(\Gamma(e))$ intersect transversely along a homoclinic orbit $\Omega(e)$ (see Definition 2.1). Therefore, there exists a 3-dimensional section Σ in \mathbb{A}^2 , transverse to X^{C_U} , such that the union $\cup_{e \in I^*} \Omega(e)$ intersects Σ along a $C^{\kappa-1}$ curve σ .

1. Since Σ is transverse to X^{C_U} , for $e \in I^*$ the intersection $\Sigma \cap C_U^{-1}(e)$ is symplectic. Reducing Σ if necessary, one easily proves the existence of a ball $B = [-\delta, \delta]^2 \subset \mathbb{R}^2$ centered at 0 and a $C^{\kappa-1}$ diffeomorphism $\chi_0 : I^* \times B \rightarrow \Sigma$, such that

- $C_U \circ \chi_0(e, s, u) = e$;
- the connected component of $W^u(A) \cap \Sigma$ containing σ admits the equation $s = 0$;
- the connected component of $W^s(A) \cap \Sigma$ containing σ admits the equation $u = 0$;
- for each $e \in I^*$, $\chi_0(e, \cdot)$ is symplectic for the usual structure on B and the induced structure on $\Sigma \cap C_U^{-1}(e)$.

2. For $\tau_0 > 0$ small enough, the Hamiltonian flow $\Phi^{C_U} :]-\tau_0, \tau_0[\times \Sigma \rightarrow \mathbb{A}^2$ is a diffeomorphism onto its image \mathcal{O} . One easily checks that one can choose the previous coordinates (u, s) in such a way that the map

$$\chi :]-\tau_0, \tau_0[\times I^* \times B \longrightarrow \mathcal{O}$$

$$\chi(\tau, e, s, u) = \Phi_\tau^{C_U}(\chi_0(e, s, u)).$$

is a $C^{\kappa-1}$ symplectic diffeomorphism. By construction, the Hamiltonian C_U takes the simple expression

$$C_U \circ \chi(\tau, e, s, u) = e.$$

This in turn yields a $C^{\kappa-1}$ symplectic diffeomorphism $\hat{\chi} : \mathcal{D} \rightarrow \mathbb{A} \times \mathcal{O} \subset \mathbb{A}^3$, where \mathcal{D} is the subset of all $(\tau, \mathbf{e}, s, u, \theta_1, r_1) \in]-\tau_0, \tau_0[\times \mathbb{R} \times B \times \mathbb{A}$ such that $\mathbf{e} - \frac{1}{2}r_1^2 \in I^*$ (note that now \mathbf{e} stands for the *total* energy of the system), defined by

$$\hat{\chi}(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \Phi_\tau^{\mathcal{H}}\left((\theta_1, r_1), \chi\left(0, \mathbf{e} - \frac{1}{2}r_1^2, s, u\right)\right) = \left((\theta_1 + \tau r_1, r_1), \chi\left(\tau, \mathbf{e} - \frac{1}{2}r_1^2, s, u\right)\right),$$

which clearly satisfies

$$\mathcal{H} \circ \widehat{\chi}(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \mathbf{e}.$$

We will set

$$\widehat{\Sigma} = \widehat{\chi}(\{(0, \frac{1}{2})\} \times \Sigma) \subset \mathcal{H}^{-1}(\mathbf{e})$$

so that \mathcal{D} is a neighborhood of $\widehat{\Sigma}$ in \mathbb{A}^3 .

3. The effect of the rescaling (9) is immediately computed in the previous straightening coordinates. We set

$$\widehat{\chi}_n = \sigma_{\sqrt{n}}^{-1} \circ \widehat{\chi}, \quad \widehat{\Sigma}_n = \sigma_{\sqrt{n}}^{-1}(\widehat{\Sigma}). \quad (14)$$

Since for t small enough

$$\widehat{\chi}^{-1} \circ \Phi_t^{\mathcal{H}} \circ \widehat{\chi}(\tau, \mathbf{e}, s, u, \theta_1, r_1) = (\tau + t, \mathbf{e}, s, u, \theta_1, r_1) \quad (15)$$

one immediately gets

$$\widehat{\chi}_n^{-1} \circ \Phi_t^{\mathcal{H}_n} \circ \widehat{\chi}_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = (\tau + \frac{1}{n}t, \mathbf{e}, s, u, \theta_1, r_1), \quad (16)$$

so that

$$\mathcal{H}_n \circ \widehat{\chi}_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \frac{\mathbf{e}}{n}.$$

4.3 Perturbation and Condition (T) for cylinders

In this section we fix $n \geq 1$ and we work with the Hamiltonian \mathcal{H}_n . We now construct a first perturbation $f_n^{(1)}$ which produces heteroclinic connections between nearby elements of $\text{Tori}(\mathcal{H}_n + f_n^{(1)})$ contained in the same cylinder and yields Condition (T) for each cylinder.

To begin with, let us consider a regular cylinder $\mathcal{C} \in \text{Cyl}(\mathcal{H}_n)$, associated with some annulus \mathbf{A} of C_U defined over an interval I , and let I^* be a subinterval of I as in the previous section. Let $F : \mathbb{T}^2 \times I \rightarrow \mathcal{H}_n^{-1}(\frac{1}{2})$ be the associated embedding of \mathcal{C} (where we assume that the natural parameter is the energy, the case of the annulus \mathbf{A}_0 is easily treated using similar arguments). As usual, we set $\mathcal{T}_e = F(\mathbb{T}^2 \times \{e\})$.

We will use the previous flow box coordinates to construct the perturbation. With the assumptions and notation of the previous section, we define a function f by

$$\widetilde{f} \circ \widehat{\chi}_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \mu \eta_\tau(\tau) \eta_\theta(\theta_1), \quad (17)$$

where $\mu > 0$ is a small enough constant and η_τ is a (nonzero) C^∞ bump function whose support is located in $[-\tau_0, 0]$ and which takes its values in $[0, 1]$, while $\eta_\theta \in C^\infty(\mathbb{T}, [0, 1])$ is a smooth function with support in $[-1/4, 1/4]$ whose derivative vanishes only at 0 in the interval $] -1/4, 1/4[$. In particular the support of f is contained in the domain $\widehat{\chi}_n(\mathcal{D})$ for coherence.

Lemma 4.7. *Fix a regular cylinder $\mathcal{C} \in \text{Cyl}(\mathcal{H}_n)$, let f be as in (17) and set $H_n = \mathcal{H}_n + f$, with $f = \widetilde{f} \circ \widehat{\chi}_n^{-1}$. Then there exist $\mu > 0$, η_τ in $C^\infty(\mathbb{R}, [0, 1])$, η_θ in $C^\infty(\mathbb{T}, [0, 1])$ and $\rho > 0$ such that the pair of invariant tori $(\mathcal{T}_e, \mathcal{T}_{e'})$ for H_n satisfies the weak splitting condition for $e, e' \in I^*$ with $|e - e'| < \rho$, and such that moreover*

$$\|f\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \nu.$$

Proof. We have fixed the energy $\mathbf{e} = \frac{1}{2}$. We first choose $\mu > 0$ small enough so that the composition $f = \tilde{f} \circ \hat{\chi}_n^{-1}$ (which is of class $C^{\kappa-1}$ since $\hat{\chi}_n$ is) satisfies $\|f\|_{C^{\kappa-1}} \leq \nu$, which is obviously possible. The coordinates $(\tau, s, u, \theta_1, r_1)$ form, via the symplectic diffeomorphism $\hat{\chi}_n$, a chart in the neighborhood of $\hat{\Sigma}_n$ in $H_n^{-1}(\frac{1}{2})$. In this chart, the vector field generated by $H_n \circ \hat{\chi}_n$ reads:

$$\begin{cases} \dot{\tau} = \frac{1}{n} \\ \dot{s} = 0 \\ \dot{u} = 0 \\ \dot{\theta}_1 = 0 \\ \dot{r}_1 = \mu \eta_\tau(\tau) \eta'_\theta(\theta_1) \end{cases} \quad (18)$$

Therefore, the variation of the variable r_1 when passing through the support of the function f is easily computed:

$$\Delta r_1 = \mu \|\eta_\tau\|_1 \eta'_\theta(\theta_1),$$

where $\|\eta_\tau\|_1 \neq 0$ is the L^1 norm. The conclusion for the existence of transverse heteroclinic intersections now easily follows: in the chart $(\tau, s, u, \theta_1, r_1)$ the intersection $\hat{\Sigma} \cap W^s(\mathcal{T}(e'))$ reads

$$\left\{ \left(0, s, 0, \theta_1, \sqrt{\frac{1}{2} - e'} \right) \mid |s| \leq \delta, \theta_1 \in \mathbb{T} \right\} \quad (19)$$

while the intersection $\hat{\Sigma} \cap W^u(\mathcal{T}(e))$ takes the form

$$\left\{ \left(0, 0, u, \theta_1, \sqrt{\frac{1}{2} - e} + \mu \|\eta_\tau\|_1 \eta'_\theta(\theta_1) \right) \mid |u| \leq \delta, \theta_1 \in \mathbb{T} \right\}. \quad (20)$$

Consider a point $\theta_1^0 \in \mathbb{T}$ and let $a \in \hat{\Sigma} \cap W^u(\mathcal{T}(e))$ be the point of coordinates

$$\left(0, 0, 0, \theta_1^0, \sqrt{\frac{1}{2} - e} + \mu \|\eta_\tau\|_1 \eta'_\theta(\theta_1^0) \right)$$

in the chart $(\tau, s, u, \theta_1, r_1)$, and let $a^\alpha \in \mathcal{T}(e)$ be its α -limit point under the flow of H_n . Then

$$a \in \Sigma \cap W^u(\mathcal{T}(e)) \cap W^s(\mathcal{T}(e')),$$

provided that

$$e' = \frac{1}{2} - \left(\sqrt{\frac{1}{2} - e} + \mu \|\eta_\tau\|_1 \eta'_\theta(\theta_1^0) \right)^2. \quad (21)$$

Now, due to the form of the vector field (18), one easily checks (using the global invariance of the unstable foliation of $W^u(\mathcal{C})$ under the flow $\Phi_t^{H_n}$) that if γ is the tangent vector to $W^{uu}(a^\alpha)$ at the point a , then the e component of γ in the flow box coordinates is nonzero as soon as θ_1 is in the support of η_θ . So the tori $\mathcal{T}(e)$ and $\mathcal{T}(e')$ admit a heteroclinic connection as soon as there exists θ_1^0 satisfying (21), which is obviously true when $|e - e'|$ is small enough (depending on μ and on the variation of η'_θ), and the previous remark moreover proves that the weak splitting condition is satisfied. \square

It remains to examine the case of the singular cylinders. The previous lemma still applies to the three regular subannuli $A_\bullet^>$, $A_\bullet^<$ and A_\bullet^0 , so it only remains to consider the neighborhood of the critical energy \hat{e} .

Lemma 4.8. *Let \mathcal{C} be one of the two singular cylinders \mathcal{C}_\bullet^\pm . Let $\nu > 0$ be fixed. Then there exists an open interval I^* containing \hat{e} and $f \in C^{\kappa-1}(\mathbb{A}^3)$*

$$\|f\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \nu$$

such that each pair $(\mathcal{T}^<(e), \mathcal{T}^0(e'))$ or $(\mathcal{T}^>(e), \mathcal{T}^0(e'))$ with $e \in I^$ and $e' \in I^*$ satisfies the weak splitting condition for the system $H_n = \mathcal{H}_n + f$.*

Proof. In fact the same considerations as in the previous lemma apply, thanks to the existence of transverse homoclinic connections for the homoclinic orbits to the fixed point O of C_U (see Definition 2.2). The singular annulus therefore admits a C^1 transverse homoclinic submanifolds $S^>, S^<$ in the neighborhood of each of the previous homoclinic connections. These submanifolds are almost everywhere of class $C^{\kappa-1}$. This enables one to find an interval I^* and associated C^1 sections $\Sigma^>, \Sigma^<$ (almost everywhere of class $C^{\kappa-1}$) as above. One readily sees that the “singular C^1 locus” causes no trouble and the same arguments as above yield the existence of f , with controlled $C^{\kappa-1}$ norm, for which the system H_n satisfies our claim. \square

Corollary 4.9. *Given $n \geq 1$, there exists $f_n^{(1)} \in C^{\kappa-1}(\mathbb{A}^3)$, with $\|f_n^{(1)}\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \frac{1}{2n}$, such that each $\mathcal{C} \in \text{Cyl}(\mathcal{H}_n + f_n^{(1)})$ satisfies Condition (T).*

Proof. We apply the previous lemma inductively, after a preliminary ordering of all subintervals $(I_*(k))_{1 \leq k \leq k^*}$ attached with all regular cylinders in $\text{Cyl}(\mathcal{H}_n)$ and a choice of pairwise disjoint attached sections Σ and homoclinic curves σ (which is obviously possible thanks to the structure of the set of annuli of C_U). Using the possibility to choose the support of the function f in Lemma 4.7 inside an arbitrary neighborhood of σ , we can therefore obtain a finite family of perturbations f^k , $1 \leq k \leq k^*$, with pairwise disjoint supports, such that the sum $f_n^{(1)} = \sum_k f^k$ satisfies the two claims of our statement since its norm is just the supremum of the individual norms. \square

4.4 Perturbation and Condition (S) for chains

So far we have constructed a perturbed Hamiltonian $\mathcal{H}_n + f_n^{(1)}$ such that each cylinder of the family $\text{Cyl}(\mathcal{H}_n + f_n^{(1)})$ satisfies Condition (T). It remains now to add a new (and smaller) perturbation term to ensure that the pairs of tori located in consecutive cylinders of the associated chain satisfy the weak splitting condition. We begin with a classical lemma on the existence of heteroclinic intersections for tori with the same homology.

Lemma 4.10. *Set $\text{Chain}(\mathcal{H}_n + f_n^{(1)}) = (\mathcal{C}_k)_{1 \leq k \leq k^*}$. Then for $1 \leq k \leq k^* - 1$, there are tori $\mathcal{T}_k \subset \mathcal{C}_k$ and $\mathcal{T}_{k+1} \subset \mathcal{C}_{k+1}$ (of the family $\text{Tori}(\mathcal{H}_n + f_n^{(1)})$) which admit a heteroclinic connection.*

Proof. Let us begin with the unperturbed situation generated by \mathcal{H}_n . Fix two consecutive (regular or singular) cylinders \mathcal{C}_k and \mathcal{C}_{k+1} , associated with annuli A_k and A_{k+1} . Then there exists an energy e for which the periodic orbits $A_k \cap C_U^{-1}(e)$ and $A_{k+1} \cap C_U^{-1}(e)$ admit a transverse heteroclinic orbit, and this situation persists in a neighborhood of e (recall that energy intervals over which our annuli are defined admit small overlapping domains). As a consequence, as above, there exists a transverse section $\hat{\Sigma} \subset \mathcal{H}_n^{-1}(\frac{1}{2})$, endowed with

symplectic coordinates (s, u, θ_1, r_1) , such that $W^u(\mathcal{C}_k) \cap \widehat{\Sigma}$ and $W^s(\mathcal{C}_{k+1}) \cap \widehat{\Sigma}$ read $\{u = 0\}$ and $\{s = 0\}$. The subset $\{u = s = 0\}$ is the (local) intersection with Σ of a manifold of heteroclinic orbits between \mathcal{C}_k and \mathcal{C}_{k+1} . This manifold \mathcal{A} is symplectic and diffeomorphic to $\mathbb{T} \times I$, where I is some (small) open interval. The invariant manifolds $W^u(\mathcal{T}_k(e))$ and $W^s(\mathcal{T}_{k+1}(e))$ intersect \mathcal{A} along an essential circle $\{r_1 = \sqrt{2(\frac{1}{2} - e)}\}$.

Now for n large enough the perturbed situation for $\mathcal{H}_n + f_n^{(1)}$ is only a slight distortion of the previous one. One can still find a section Σ with coordinates (s, u, θ_1, r_1) in which $W^u(\mathcal{C}_k) \cap \Sigma$ and $W^s(\mathcal{C}_{k+1}) \cap \Sigma$ have the same equations as above and so intersect along the slightly perturbed annulus \mathcal{A}' with equation $u = s = 0$ in the new coordinates (all this being deduced from the various transversality properties). Again, by transversality, $W^u(\mathcal{T}_k(e)) \cap \mathcal{A}$ and $W^s(\mathcal{T}_{k+1}(e)) \cap \mathcal{A}$ are embedded essential circles but they do not coincide any longer (in general).

However, it is easy to see that they still intersect each other, using the fact that the coordinates (θ_1, r_1) are exact symplectic on \mathcal{A} together with the Lagrangian character of the invariant manifolds $W^u(\mathcal{T}_k(e))$ and $W^s(\mathcal{T}_{k+1}(e))$ (see [LMS03] for more details). Indeed, since the tori $\mathcal{T}_k(e)$ and $\mathcal{T}_{k+1}(e)$ are left unchanged when the perturbation is added, the intersections $C_k = W^u(\mathcal{T}_k(e)) \cap \mathcal{A}$ and $C_{k+1} = W^s(\mathcal{T}_{k+1}(e)) \cap \mathcal{A}$ have the same homology in \mathcal{A}' , meaning that the symplectic area between them vanishes. This comes from the fact that this assertion is trivially true in the unperturbed situation along with the Lagrangian character of $W^u(\mathcal{T}_k(e))$ and $W^s(\mathcal{T}_{k+1}(e))$. This proves our claim. \square

Our next lemma will enable us to complete the proof of Proposition 4.6

Lemma 4.11. *For $n \geq n_0$ large enough, there exists a function $f_n \in C^{\kappa-1}(\mathbb{A}^3)$ with support contained in the complement of $\cup_{1 \leq k \leq k^*} \mathcal{C}_k$, with $\|f_n\|_{C^{\kappa-1}(\mathbb{A}^3)} \leq \frac{1}{n}$, such that the chain $(\mathcal{C}_k)_{1 \leq k \leq k^*}$ for the system $H_n := \mathcal{H}_n + f_n$ satisfies Condition (S).*

Proof. The proof is similar and even simpler than that of Lemma 4.7. With the notation of Lemma 4.10, if the circles C_k and C_{k+1} intersect transversely in \mathcal{A} , there is obviously nothing to do. Now if they intersect tangentially, one constructs a flow-box as in Section 4.2 and again uses a perturbation of the form

$$\ell_n \circ \widehat{\chi}_n(\tau, \mathbf{e}, s, u, \theta_1, r_1) = \mu \eta_\tau(\tau) \eta_\theta(\theta_1).$$

The support of ℓ_n can be chosen arbitrarily small, and its norm is controlled by means of the constant μ . In particular, it can be chosen small enough to preserve the Condition (T) for all cylinders. One can therefore proceed by induction as above, using now the natural ordering of the heteroclinically connected pairs of tori inside consecutive cylinders of the chain. This proves the existence of a finite family of functions ℓ_n^j , with controlled supports and norms, such that $f_n = f_n^{(1)} + \sum_j \ell_n^j$ fulfills our claims. \square

5 Diffusion orbits and proof of Theorem 1.1

We first recall the λ -lemma of [S13] in a version adapted to our present setting and state an abstract shadowing result for chains of cylinders. We then apply this result to prove the main theorem of this paper.

5.1 Shadowing orbits along chains of minimal sets

The λ -lemma in [S13] requires the existence of the “straightening neighborhood” (Proposition B) for the cylinders. In the case of general normally hyperbolic manifolds such results need abstract additional assumptions, but here we will take advantage of the very simple geometric structure of the problem.

1. Let us begin with a straightening result in the neighborhood of the annuli. Let $U \in \mathcal{U}$ be fixed.

Lemma 5.1. *Let A be an annulus defined over I for C_U . Then there exist a neighborhood \mathcal{O} of A , an interval \widehat{I} containing I and a symplectic diffeomorphism $\Psi : \mathbb{T} \times \widehat{I} \times B \rightarrow \mathcal{O}$, where $B = [-\alpha, \alpha]^2$ is a ball in \mathbb{R}^2 , such that $A = \Psi(\mathbb{T} \times I \times \{0\})$ and the composed Hamiltonian $C = C_U \circ \Psi$ takes the form*

$$C(\varphi, \rho, u, s) = C_0(\rho) + \lambda(\varphi, \rho)us + C_3(\varphi, \rho, u, s) \quad (22)$$

with

$$C_3(\varphi, \rho, 0, 0) = 0, \quad DC_3(\varphi, \rho, 0, 0) = 0, \quad D^2C_3(\varphi, \rho, 0, 0) = 0. \quad (23)$$

In particular, the local stable and unstable manifolds $W_\ell^{s,u}(A)$ together with the local stable and unstable manifolds $W_\ell^{ss,uu}(x)$ for $x \in A$ are straightened in these coordinates and read:

$$\Psi^{-1}(W_\ell^s(A)) = \{u = 0\}, \quad \Psi^{-1}(W_\ell^u(A)) = \{s = 0\},$$

$$\Psi^{-1}(W_\ell^{ss}(x)) = \{(\varphi, \rho, s, 0) \mid s \in [-\alpha, \alpha]\}, \quad \Psi^{-1}(W_\ell^{uu}(x)) = \{(\varphi, \rho, 0, u) \mid u \in [-\alpha, \alpha]\},$$

where (φ, ρ) is defined by $\Psi(x) = (\varphi, \rho, 0, 0)$.

The proof is a simple application of the Moser isotopy lemma. One proves indeed the straightening result first and deduces the normal form from the structure of the Hamiltonian system in such a neighborhood. The previous lemma yields the following straightening result in the neighborhood of the cylinders of $\text{Cyl}(H_n)$.

Lemma 5.2. *Let \mathcal{C} be a cylinder of the family $\text{Cyl}(H_n)$ and let A be the associated annulus, defined over I . Let \mathcal{O} and Ψ be defined as in the previous lemma. Then, up to shrinking B if necessary, the product diffeomorphism*

$$\widehat{\Psi} = \text{Id}_{\mathbb{A}} \times \Psi : \mathbb{A} \times \mathbb{T} \times \widehat{I} \times B \longrightarrow \mathbb{A} \times \mathcal{O}$$

is symplectic and satisfies

$$H_n \circ \widehat{\Psi}(\theta_1, r_1, \varphi, \rho, s, u) = \frac{1}{2}r_1^2 + C_0(\rho) + O_2(s, u).$$

Proof. This is an immediate consequence of the fact that if B is small enough, the neighborhood $\mathbb{A} \times \mathcal{O}$ and the support of f_n are disjoint, so that $(H_n)_{\mathbb{A} \times \mathcal{O}} = (\mathcal{H}_n)_{\mathbb{A} \times \mathcal{O}}$. The claim then follows from the previous lemma. \square

Note that \mathcal{C} is then the set of all $\widehat{\Psi}(\theta_1, r_1, \varphi, \rho, 0, 0)$ such that

$$\frac{1}{2}r_1^2 + C_0(\rho) = \frac{1}{2}.$$

The λ -lemma proved in [S13] was stated in the framework of symplectic diffeomorphisms and normally hyperbolic invariant submanifolds in a symplectic manifold. We therefore need to adapt it to the present context, since the cylinders \mathcal{C} are not normally hyperbolic in \mathbb{A}^3 , but rather in $H_n^{-1}(\frac{1}{2})$. The simplest way to overcome this (easy) problem is to apply the lemma to the full normally hyperbolic manifold $\mathcal{N} = \widehat{\Psi}(\mathbb{A} \times \mathbb{T} \times \widehat{I} \times \{0\})$ (with the notation of the previous lemma) and the symplectic diffeomorphism Φ^{H_n} (the time-one map). This is made possible by the previous straightening result (see [S13] for a proof, the lack of compactness obviously causes no trouble here, due to the preservation of energy and the fact that \mathcal{C} is relatively compact). We set $\Phi = \Phi^{H_n}$.

The λ -lemma. *Let $\mathcal{C} \in \text{Cyl}(H_n)$ be a cylinder at energy $\frac{1}{2}$ for the Hamiltonian system H_n and let \mathcal{N} be the normally hyperbolic manifold of \mathbb{A}^3 defined above. Let Δ be a 1-dimensional submanifold of \mathbb{A}^3 which transversely intersects $W^s(\mathcal{N})$ at some point a . Then $\Phi^n(\Delta)$ converges to the unstable leaf $W^{uu}(\Phi^n(\ell(a)))$ in the C^0 compact open topology, where $\ell(a)$ is the unique element of \mathcal{C} such that the point a belongs to the stable leaf $W^{ss}(\ell(a))$.*

Let us make clear the notion of convergence used here (see [S13] for details). The simplest way to define it is to use Lemma 5.2. In the neighborhood $\mathbb{A} \times \mathcal{O}$ and relatively to the previous coordinates, if $x \sim (\theta_1, r_1, \varphi, \rho, 0, 0) \in \mathcal{C}$, the unstable leaf $W^{uu}(x)$ reads

$$W^{uu}(x) = \{(\theta_1, r_1, \varphi, \rho, 0, u) \mid u \in [-\alpha, \alpha]\}.$$

The first result in [S13] (Theorem 1) is that for n large enough, the connected component Δ^n of $\Phi^n(a)$ in $\Phi(\Delta^{n-1}) \cap (\mathbb{A} \times \mathcal{O})$ is a *graph* over the unstable direction, that is, it admits the equation

$$\Delta^n = \left\{ (\theta_1^n(u), r_1^n(u), \varphi^n(u), \rho^n(u), s^n(u), u) \mid u \in]-\overline{u}, \overline{u}[\right\}.$$

The convergence statement then just says that

$$\|(\theta_1^n(u), r_1^n(u), \varphi^n(u), \rho^n(u), s^n(u)) - (\theta_1^n(0), r_1^n(0), \varphi^n(0), \rho^n(0), 0)\| \rightarrow 0$$

uniformly in u when n tends to $+\infty$, where $(\theta_1^n(0), r_1^n(0), \varphi^n(0), \rho^n(0), 0, 0) \sim \Phi^n(x)$. Of course one then gets more global formulation by using the definition of $W^u(\mathcal{C})$ as the union of the images by Φ of the local unstable manifold. Note that this is only a C^0 -convergence while a stronger C^1 -convergence result was proved in [S13]. The same definitions apply to the following case.

Corollary 5.3. *Let $\mathcal{C} \in \text{Cyl}(H_n)$. Let Δ be a 1-dimensional submanifold of $H_n^{-1}(\frac{1}{2})$ which transversely intersects $W^s(\mathcal{C})$ in $H_n^{-1}(\frac{1}{2})$ at some point a . Then $\Phi^n(\Delta)$ converges to the unstable leaf $W^{uu}(\Phi^n(\ell(a)))$ in the C^0 compact open topology.*

Proof. Observe that the fact that Δ intersects $W^s(\mathcal{C})$ transversely in $H_n^{-1}(\frac{1}{2})$ implies that Δ transversely intersects $W^s(\mathcal{N})$. Then apply the λ -lemma and use the invariance of energy. \square

2. We can now state the shadowing result proved in [S13]. The method is reminiscent of that of [BT99].

Proposition 5.4. [Shadowing lemma]. *Set $\text{Chain}(H_n) = (\mathcal{C}^i)_{1 \leq i \leq i^*(n)}$. For $1 \leq i \leq i^*$, let $(\mathcal{T}_j^i)_{1 \leq j \leq j_i^*}$ be a family of dynamically minimal invariant tori contained in \mathcal{C}^i , such that*

- *for $1 \leq j \leq j_i^* - 1$, there exists $a_j^i \in \mathcal{T}_j^i$ such that $W^{uu}(a_j^i)$ intersects $W^s(\mathcal{C}^i)$ transversely in $H_n^{-1}(\frac{1}{2})$, at some point contained in $W^s(\mathcal{T}_{j+1}^i)$,*
- *for $1 \leq i \leq i^* - 1$, there exists $a_{j_i^*}^i \in \mathcal{T}_{j_i^*}^i$ such that $W^{uu}(a_{j_i^*}^i)$ intersects $W^s(\mathcal{C}^{i+1})$ transversely in $H_n^{-1}(\frac{1}{2})$, at some point contained in $W^s(\mathcal{T}_1^{i+1})$.*

Then, for each $\rho > 0$, there exists an orbit Γ at energy $\frac{1}{2}$ of H_n which intersects each ρ -neighborhood $\mathcal{V}_\rho(\mathcal{T}_j^i)$, for $1 \leq i \leq i^$ and $1 \leq j \leq j_i^*$.*

5.2 Asymptotic density: proof of Theorem 1.1

It only remains now to gather the results of the previous sections and apply the previous shadowing lemma to the chain of cylinders $\text{Chain}(H_n)$ and a suitable family of minimal tori inside. Fix $\delta > 0$. Given $n \geq 1$, we set as above $\text{Chain}(H_n) = (\mathcal{C}^i)_{1 \leq i \leq i^*(n)}$.

- There exists n_0 such that for $n \geq n_0$, the union of the lines $(\mathbb{S}_{k_\ell})_{1 \leq \ell \leq n}$ is $\delta/4$ -dense in \mathbb{S} .
- There exists $n_1 \geq n_0$ such that for $n \geq n_1$, the sphere \mathbb{S} is $\delta/4$ -dense in $H_n^{-1}(\frac{1}{2})$, and therefore the union of the lines $(\mathbb{S}_{k_\ell})_{1 \leq \ell \leq n}$ is $\delta/2$ -dense in $H_n^{-1}(\frac{1}{2})$.
- By construction, there exists $n_2 \geq n_1$ such that for $n \geq n_2$,

$$\mathbf{d}\left(\bigcup_{1 \leq i \leq i^*(n)} \Pi(\mathcal{C}^i), \bigcup_{1 \leq \ell \leq n} \mathbb{S}_{k_\ell}\right) \leq \delta/6,$$

where \mathbf{d} is the Hausdorff distance in \mathbb{R}^3 . This shows that $\bigcup_{1 \leq i \leq i^*(n)} \Pi(\mathcal{C}^i)$ is $\delta/6$ -dense in $\bigcup_{1 \leq \ell \leq n} \mathbb{S}_{k_\ell}$.

• By density of the minimal tori in the cylinders (see Proposition 3.4), and since the chains satisfy Condition (S), for each $n \geq n_2$, one can exhibit a family of minimal tori (\mathcal{T}_j^i) satisfying the assumptions of the shadowing lemma and such that the union $\bigcup_{i,j} \Pi(\mathcal{T}_j^i)$ is $\delta/6$ -dense in $\bigcup_{1 \leq i \leq i^*(n)} \Pi(\mathcal{C}^i)$.

• Proposition 5.4, applied with $\rho = \delta/6$, shows the existence of an orbit of H_n whose projection is $\delta/6$ -dense in $\bigcup_{i,j} \Pi(\mathcal{T}_j^i)$ and therefore $\delta/2$ -dense in $\bigcup_{1 \leq \ell \leq n} \mathbb{S}_{k_\ell}$, so also δ -dense in $H_n^{-1}(\frac{1}{2})$. This concludes the proof of Theorem 1.1.

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